MATH 2028 Honours Advanced Calculus II
2021-22 Term 1
Problem Set 11
due on Dec 6, 2021 (Monday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Gradescope on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Problems to hand in

1. Compute the area of the surface in $\mathbb{R}^4$ parametrized by
$$g(u,v) = (u, v, u^2 - v^2, 2uv)$$
with $(u,v) \in \mathbb{R}^2$ satisfying $u^2 + v^2 \leq 1$.

2. Let $\Omega \subset \mathbb{R}^3$ be the region bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the plane $z = 0$. Compute
$$\int_{\partial\Omega} xz \; dy \wedge dz + yz \; dz \wedge dx + (x^2 + y^2 + z^2) \; dx \wedge dy$$
directly and by applying Stokes' Theorem.

3. (a) Suppose M and M' are two compact oriented k-dimensional submanifolds of $\mathbb{R}^n$ with boundary, and suppose $\partial M = \partial M'$. Prove that for any $(k-1)$ form ω, we have
$$\int_M d\omega = \int_{M'} d\omega.$$
(b) Use (a) to compute $\int_M d\omega$ where M is the upper hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, oriented with outward-pointing normal having positive z-component and
$$\omega = (x^3 + 3x^2 y - y) \; dx + (y^3 z + x + x^3) \; dy + (x^2 + y^2 + z) \; dz.$$

Suggested Exercises

1. Check that the boundary orientation on $\partial\mathbb{R}^k_+$ is $(-1)^k$ times the usual orientation on $\mathbb{R}^{k-1}$.

2. Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $2x + 3y - z = 1$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate
$$\int_C y \; dx - 2z \; dy + x \; dz$$
directly and by applying Stokes' Theorem.

3. Compute $\int_C (y - z) \; dx + (z - x) \; dy + (x - y) \; dz$ where C is the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $\frac{x}{a} + \frac{z}{b} = 1$, oriented clockwise as viewed from high above the xy-plane.

4. Let C be the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + y + z = 0$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate

$$\int_C 2z\ dx + 3x\ dy - dz.$$

5. Let $\omega = y^2\ dy \wedge dz + x^2\ dz \wedge dx + z^2\ dx \wedge dy$, and M be the solid paraboloid $0 \leq z \leq 1 - x^2 - y^2$. Evaluate $\int_{\partial M} \omega$ directly and by applying Stokes' Theorem.

6. Let M be the surface of the paraboloid $z = 1 - x^2 - y^2 \geq 0$, oriented so that the outward-pointing normal has positive z-component. Let $F(x, y, z) = (x^2 z, y^2 z, x^2 + y^2)$. Compute $\int_M F \cdot \vec{n}\ d\sigma$ directly and by applying Stokes' Theorem.

7. Compute $\int_M d\omega$ where M is the portion of the paraboloid $z = x^2 + y^2$ lying beneath $z = 4$, oriented with outward-pointing normal having positive z-component, and $\omega = y\ dx + z\ dy + x\ dz$.

8. Let $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4\ :\ x_1^2 + x_2^2 + x_3^2 \leq x_4 \leq 1\}$, with the standard orientation inherited from $\mathbb{R}^4$. Evaluate

$$\int_{\partial M} (x_1^3 x_2^4 + x_4)\ dx_1 \wedge dx_2 \wedge dx_3.$$

9. Let S be the portion of the cylinder $x^2 + y^2 = a^2$ lying above the xy-plane and below the sphere $x^2 + (y - a)^2 + z^2 = 4a^2$. Let C be the intersection of the cylinder and sphere, oriented clockwise as viewed from high above the xy-plane.

   (a) Evaluate $\int_S z\ d\sigma$.

   (b) Use (a) to compute $\int_C y(z^2 - 1)\ dx + x(1 - z^2)\ dy + z^2\ dz$.

10. Let C be the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate $\int_C z^3\ ds$.

Challenging Exercises

1. Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function whose graph is the surface S.

   (a) Consider the area 2-form σ on S given by

   $$\sigma = \frac{1}{\sqrt{1 + |\nabla f|^2}} \left(-\frac{\partial f}{\partial x}\ dy \wedge dz - \frac{\partial f}{\partial y}\ dz \wedge dx + dx \wedge dy \right).$$

   Show that $d\sigma = 0$ if and only if f satisfies the minimal surface equation:

   $$\left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right) \frac{\partial^2 f}{\partial x^2} - 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial y} + \left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right) \frac{\partial^2 f}{\partial y^2} = 0.$$

   (b) Show that for any compact oriented surface $N \subset \mathbb{R}^3$, we have

   $$\int_N \sigma \leq \text{area}(N)$$

   and equality holds if and only if N is parallel to S.

(c) Suppose further that $\partial N = \partial S$. Prove that $\text{area}(S) \leq \text{area}(N)$.

2. (a) Prove that a k-dimensional submanifold with boundary $M \subset \mathbb{R}^n$ is orientable if and only if there is a nowhere-zero k-form on M.

   (b) Show that M is orientable if and only if there is a volume form globally defined on M.